

Contents

Preface	v
Notations	xv

Part 1. Simplicial Techniques and Homology

Introduction	3
Chapter I. The Work of Poincaré	15
§ 1. Introduction	15
§ 2. Poincaré's First Paper: <i>Analysis Situs</i>	16
§ 3. Heegaard's Criticisms and the First Two <i>Compléments à l'Analysis Situs</i>	28
Chapter II. The Build-Up of “Classical” Homology	36
§ 1. The Successors of Poincaré	36
§ 2. The Evolution of Basic Concepts and Problems	37
§ 3. The Invariance Problem	43
§ 4. Duality and Intersection Theory on Manifolds	49
A. The Notion of “Manifold”	49
B. Computation of Homology by Blocks	50
C. Poincaré Duality for Combinatorial Manifolds	50
D. Intersection Theory for Combinatorial Manifolds	51
§ 5. Homology of Products of Cell Complexes	55
§ 6. Alexander Duality and Relative Homology	56
Chapter III. The Beginnings of Differential Topology	60
§ 1. Global Properties of Differential Manifolds	60
§ 2. The Triangulation of C^1 Manifolds	62
§ 3. The Theorems of de Rham	62
Chapter IV. The Various Homology and Cohomology Theories	67
§ 1. Introduction	67
§ 2. Singular Homology versus Čech Homology and the Concept of Duality	68
§ 3. Cohomology	78
§ 4. Products in Cohomology	81
A. The Cup Product	81
B. The Functional Cup Product	84
C. The Cap Product	84

§ 5. The Growth of Algebraic Machinery and the Forerunners of Homological Algebra	85
A. Exact Sequences	85
B. The Functors \otimes and Tor	90
C. The Künneth Formula for Chain Complexes	92
D. The Functors Hom and Ext	93
E. The Birth of Categories and Functors	96
F. Chain Homotopies and Chain Equivalences	98
G. Acyclic Models and the Eilenberg–Zilber Theorem	100
H. Applications to Homology and Cohomology of Spaces: Cross Products and Slant Products	102
§ 6. Identifications and Axiomatizations	105
A. Comparison of Vietoris, Čech, and Alexander–Spanier Theories	105
B. The Axiomatic Theory of Homology and Cohomology	107
C. Cohomology of Smooth Manifolds	113
D. Cubical Singular Homology and Cohomology	115
E. Leray's 1945 Paper	115
§ 7. Sheaf Cohomology	120
A. Homology with Local Coefficients	120
B. The Concept of Sheaf	123
C. Sheaf Cohomology	128
D. Spectral Sequences	132
E. Applications of Spectral Sequences to Sheaf Cohomology	138
F. Coverings and Sheaf Cohomology	141
G. Borel–Moore Homology	143
§ 8. Homological Algebra and Category Theory	147
A. Homological Algebra	147
B. Dual Categories	149
C. Representable Functors	151
D. Abelian Categories	155

*Part 2. The First Applications of Simplicial Methods
and of Homology*

Introduction	161
Chapter I. The Concept of Degree	167
§ 1. The Work of Brouwer	167
§ 2. The Brouwer Degree	169
§ 3. Later Improvements and Variations	173
A. Homological Interpretation of the Degree	173
B. Order of a Point with Respect to a Hypersurface; the Kronecker Integral and the Index of a Vector Field.	175
C. Linking Coefficients	176
D. Localization of the Degree	178
§ 4. Applications of the Degree	180
Chapter II. Dimension Theory and Separation Theorems	182
§ 1. The Invariance of Dimension	182
§ 2. The Invariance of Domain	183

§ 3. The Jordan–Brouwer Theorem	185
A. Lebesgue’s Note	185
B. Brouwer’s First Paper on the Jordan–Brouwer Theorem	185
C. Brouwer’s Second Paper on the Jordan–Brouwer Theorem	187
§ 4. The No Separation Theorem	189
§ 5. The Notion of Dimension for Separable Metric Spaces	191
§ 6. Later Developments	194
 Chapter III. Fixed Points	197
§ 1. The Theorems of Brouwer	197
§ 2. The Lefschetz Formula	198
§ 3. The Index Formula	201
 Chapter IV. Local Homological Properties	204
§ 1. Local Invariants	204
A. Local Homology Groups and Local Betti Numbers	204
B. Application to the Local Degree	206
C. Later Developments	206
D. Phragmén–Brouwer Theorems and Unicoherence	207
§ 2. Homological and Cohomological Local Connectedness	208
§ 3. Duality in Manifolds and Generalized Manifolds	210
A. Fundamental Classes and Duality	210
B. Duality in Generalized Manifolds	211
 Chapter V. Quotient Spaces and Their Homology	214
§ 1. The Notion of Quotient Space	214
§ 2. Collapsing and Identifications	216
A. Collapsing	216
B. Cones	216
C. Suspension	217
D. Wedge and Smash Product	217
E. Join of Two Spaces	218
F. Doubling	218
G. Connected Sums	219
§ 3. Attachments and CW-Complexes	220
A. The Mapping Cylinder	220
B. The Mapping Cone	220
C. The CW-Complexes	221
§ 4. Applications: I. Homology of Grassmannians, Quadrics, and Stiefel Manifolds	223
A. Homology of Projective Spaces	223
B. Homology of Grassmannians	224
C. Homology of Quadrics and Stiefel Manifolds	225
§ 5. Applications: II. The Morse Inequalities	227
 Chapter VI. Homology of Groups and Homogeneous Spaces	232
§ 1. The Homology of Lie Groups	232
§ 2. H-Spaces and Hopf Algebras	234
A. Hopf’s Theorem	234
B. Samelson’s Theorem and Pontrjagin Product	238

C. Interpretation of the Rank in Cohomology	241
D. Remarks	242
§3. Action of Transformation Groups on Homology	242
A. Complexes with Automorphisms	242
B. The Franz–Reidemeister Torsion	244
C. Fixed Points of Periodic Automorphisms	246
Chapter VII. Applications of Homology to Geometry and Analysis	249
§1. Applications to Algebraic Geometry	249
A. Early Applications	249
B. The Work of Lefschetz	251
C. The Triangulation of Algebraic Varieties	253
D. The Hodge Theory	254
§2. Applications to Analysis	257
A. Fixed Point Theorems	257
B. The Leray–Schauder Degree	260
§3. The Calculus of Variations in the Large (Morse Theory)	262

Part 3. Homotopy and its Relation to Homology

Introduction	273
Chapter I. Fundamental Group and Covering Spaces	293
§1. Covering Spaces	293
§2. The Theory of Covering Spaces	296
§3. Computation of Fundamental Groups	301
A. Elementary Properties	301
B. Fundamental Groups of Simplicial Complexes	301
C. Covering Spaces of Complexes	302
D. The Seifert–van Kampen Theorem	302
E. Fundamental Group and One-Dimensional Homology Group	304
§4. Examples and Applications	305
A. Fundamental Groups of Graphs	305
B. The “Gruppenbild”	306
C. Fundamental Group of a H-Space	307
D. Poincaré Manifolds	307
E. Knots and Links	307
Chapter II. Elementary Notions and Early Results in Homotopy Theory	311
§1. The Work of H. Hopf	311
A. Brouwer’s Conjecture	311
B. The Hopf Invariant	314
C. Generalizations to Maps from S_{2k-1} into S_k	317
§2. Basic Notions in Homotopy Theory	320
A. Homotopy and Extensions	320
B. Retracts and Extensions	321
C. Homotopy Type	323
D. Retracts and Homotopy	326

E. Fixed Points and Retracts	329
F. The Lusternik–Schnirelmann Category	329
§3. Homotopy Groups	330
A. The Hurewicz Definition	330
B. Elementary Properties of Homotopy Groups	333
C. Suspensions and Loop Spaces	334
D. The Homotopy Suspension	336
E. Whitehead Products	337
F. Change of Base Points	338
§4. First Relations between Homotopy and Homology	339
A. The Hurewicz Homomorphism	339
B. Application to the Hopf Classification Problem	341
C. Obstruction Theory	342
§5. Relative Homotopy and Exact Sequences	347
A. Relative Homotopy Groups	347
B. The Exact Homotopy Sequence	350
C. Triples and Triads	352
D. The Barratt–Puppe Sequence	353
E. The Relative Hurewicz Homomorphism	355
F. The First Whitehead Theorem	355
§6. Homotopy Properties of CW-Complexes	356
A. Aspherical Spaces	356
B. The Second Whitehead Theorem	358
C. Lemmas on Homotopy in Relative CW-Complexes	360
D. The Homotopy Excision Theorem	362
E. The Freudenthal Suspension Theorems	364
F. Realizability of Homotopy Groups	366
G. Spaces Having the Homotopy Type of CW-Complexes	369
§7. Simple Homotopy Type	369
A. Formal Deformations	369
B. The Whitehead Torsion	372
Chapter III. Fibrations	385
§1. Fibers and Fiber Spaces	385
A. From Vector Fields to Fiber Spaces	385
B. The Definition of (Locally Trivial) Fiber Spaces	387
C. Basic Properties of Fiber Spaces	389
§2. Homotopy Properties of Fibrations	398
A. Covering Homotopy and Fibrations	398
B. Fiber Spaces and Fibrations	402
C. The Homotopy Exact Sequence of a Fibration	406
D. Applications to Computations of Homotopy Groups	409
E. Classifying Spaces: I. The Whitney–Steenrod Theorems	411
F. Classifying Spaces: II. Later Improvements	414
G. Classifying Spaces: III. The Milnor Construction	417
H. The Classification of Principal Fiber Spaces with Base Space S_n	419
Chapter IV. Homology of Fibrations	421
§1. Characteristic Classes	421
A. The Stiefel Classes	421

B. Whitney's Work	422
C. Pontrjagin Classes	426
D. Chern Classes	430
E. Later Results	432
§2. The Gysin Exact Sequence	435
§3. The Spectral Sequences of a Fibration	439
A. The Leray Cohomological Spectral Sequence of a Fiber Space	439
B. The Transgression	442
C. The Serre Spectral Sequences	443
§4. Applications to Principal Fiber Spaces	447
Chapter V. Sophisticated Relations between Homotopy and Homology	453
§1. Homology and Cohomology of Discrete Groups	453
A. The Second Homology Group of a Simplicial Complex	453
B. The Homology of Aspherical Simplicial Complexes	455
C. The Eilenberg Groups	457
D. Homology and Cohomology of Groups	458
E. Application to Covering Spaces	463
§2. Postnikov Towers and Eilenberg–Mac Lane Fibers	465
A. The Eilenberg–Mac Lane Invariant	465
B. The Postnikov Invariants	466
C. Fibrations with Eilenberg–Mac Lane Fibers	469
D. The Homology Suspension	472
§3. The Homology of Eilenberg–Mac Lane Spaces	474
A. The Topological Approach	474
B. The Bar Construction	475
C. The Cartan Constructions	477
§4. Serre's \mathcal{C} -Theory	478
A. Definitions	478
B. The Absolute \mathcal{C} -Isomorphism Hurewicz Theorem	481
C. The Relative \mathcal{C} -Isomorphism Hurewicz Theorem	483
D. The First Whitehead \mathcal{C} -Theorem	483
§5. The Computation of Homotopy Groups of Spheres	484
A. Serre's Finiteness Theorem for Odd-Dimensional Spheres	484
B. Serre's Finiteness Theorem for Even-Dimensional Spheres	488
C. Wedges of Spheres and Homotopy Operations	489
D. Freudenthal Suspension, Hopf Invariant, and James Exact Sequence	492
E. The Localization of Homotopy Groups	495
F. The Explicit Computation of the $\pi_{n+k}(S_n)$ for $k > 0$	495
§6. The Computation of Homotopy Groups of Compact Lie Groups	496
A. Serre's Method	496
B. Bott's Periodicity Theorems	498
C. Later Developments	508
Chapter VI. Cohomology Operations	510
§1. The Steenrod Squares	510
A. Mappings of Spheres and Cup-Products	510
B. The Construction of the Steenrod Squares	511
§2. The Steenrod Reduced Powers	515
A. New Definition of the Steenrod Squares	515

B. The Steenrod Reduced Powers: First Definition	517
C. The Steenrod Reduced Powers: Second Definition	519
§3. Cohomology Operations	523
A. Cohomology Operations and Eilenberg–Mac Lane Spaces	523
B. The Cohomology Operations of Type $(q, n, \Pi, \mathbf{F}_2)$	525
C. The Relations between the Steenrod Squares	527
D. The Relations between the Steenrod Reduced Powers, and the Steenrod Algebra	528
E. The Pontrjagin p -th Powers	532
§4. Applications of Steenrod's Squares and Reduced Powers	533
A. The Steenrod Extension Theorem	533
B. Steenrod Squares and Stiefel–Whitney Classes	537
C. Application to Homotopy Groups	541
D. Nonexistence Theorems	544
§5. Secondary Cohomology Operations	545
A. The Notion of Secondary Cohomology Operations	545
B. General Constructions	546
C. Special Secondary Cohomology Operations	548
D. The Hopf Invariant Problem	549
E. Consequences of Adams' Theorem	551
§6. Cohomotopy Groups	551
A. Cohomotopy Sets	552
B. Cohomotopy Groups	553
 Chapter VII. Generalized Homology and Cohomology	555
§1. Cobordism	555
A. The Work of Pontrjagin	555
B. Transversality	556
C. Thom's Basic Construction	558
D. Homology and Homotopy of Thom Spaces	559
E. The Realization Problem	561
F. Smooth Classes in Simplicial Complexes	563
G. Unoriented Cobordism	564
H. Oriented Cobordism	574
I. Later Developments	575
§2. First Applications of Cobordism	580
A. The Riemann–Roch–Hirzebruch Theorem	580
I. The Arithmetic Genus	580
II. The Todd Genus	581
III. Divisors and Line Bundles	582
IV. The Riemann–Roch Problem	584
V. Virtual Genus and Arithmetic Genus	585
VI. The Introduction of Sheaves	586
VII. The Sprint	587
VIII. The Grand Finale	591
B. Exotic Spheres	595
§3. The Beginnings of K-Theory	598
A. The Grothendieck Groups	598
B. Riemann–Roch Theorems for Differentiable Manifolds	601

§4. S-Duality	603
§5. Spectra and Theories of Generalized Homology and Cohomology	606
A. K-Theory and Generalized Cohomology	606
B. Spectra	607
C. Spectra and Generalized Cohomology	608
D. Generalized Homology and Stable Homotopy	610
Bibliography	612
Index of Cited Names	633
Subject Index	639