
Contents

1	A round-up on numbers	1
1.1	Mathematical induction	1
1.2	The concept of recursion	5
1.2.1	Fibonacci numbers	6
1.2.2	Further examples of population dynamics	11
1.2.3	The tower of Hanoi: a non-homogeneous linear case	13
1.3	The Euclidean algorithm	14
1.3.1	Division	14
1.3.2	The greatest common divisor	16
1.3.3	Bézout's identity	17
1.3.4	Linear Diophantine equations	20
1.3.5	Euclidean rings	21
1.3.6	Polynomials	23
1.4	Counting in different bases	30
1.4.1	Positional notation of numbers	30
1.4.2	Base 2	32
1.4.3	The four operations in base 2	33
1.4.4	Integer numbers in an arbitrary base	39
1.4.5	Representation of real numbers in an arbitrary base	40
1.5	Continued fractions	43
1.5.1	Finite simple continued fractions and rational numbers	44
1.5.2	Infinite simple continued fractions and irrational numbers	48
1.5.3	Periodic continued fractions	56
1.5.4	A geometrical model for continued fractions	57
1.5.5	The approximation of irrational numbers by convergents	58
1.5.6	Continued fractions and Diophantine equations	61
	Appendix to Chapter 1	62
A1	Theoretical exercises	62
B1	Computational exercises	73
C1	Programming exercises	84

2	Computational complexity	87
2.1	The idea of computational complexity	87
2.2	The symbol \mathcal{O}	89
2.3	Polynomial time, exponential time	92
2.4	Complexity of elementary operations	95
2.5	Algorithms and complexity	97
2.5.1	Complexity of the Euclidean algorithm	98
2.5.2	From binary to decimal representation: complexity	101
2.5.3	Complexity of operations on polynomials	101
2.5.4	A more efficient multiplication algorithm	103
2.5.5	The Ruffini–Horner method	105
	Appendix to Chapter 2	107
A2	Theoretical exercises	107
B2	Computational exercises	109
C2	Programming exercises	113
3	From infinite to finite	115
3.1	Congruence: fundamental properties	115
3.2	Elementary applications of congruence	120
3.2.1	Casting out nines	120
3.2.2	Tests of divisibility	121
3.3	Linear congruences	122
3.3.1	Powers modulo n	126
3.4	The Chinese remainder theorem	128
3.5	Examples	133
3.5.1	Perpetual calendar	133
3.5.2	Round-robin tournaments	136
	Appendix to Chapter 3	136
A3	Theoretical exercises	136
B3	Computational exercises	140
C3	Programming exercises	147
4	Finite is not enough: factoring integers	149
4.1	Prime numbers	149
4.1.1	The Fundamental Theorem of Arithmetic	150
4.1.2	The distribution of prime numbers	152
4.1.3	The sieve of Eratosthenes	157
4.2	Prime numbers and congruences	160
4.2.1	How to compute Euler function	160
4.2.2	Fermat’s little theorem	162
4.2.3	Wilson’s theorem	165
4.3	Representation of rational numbers in an arbitrary base	166
4.4	Fermat primes, Mersenne primes and perfect numbers	168
4.4.1	Factorisation of integers of the form $b^n \pm 1$	168
4.4.2	Fermat primes	170

4.4.3	Mersenne primes	172
4.4.4	Perfect numbers	173
4.5	Factorisation in an integral domain	173
4.5.1	Prime and irreducible elements in a ring	174
4.5.2	Factorial domains	175
4.5.3	Noetherian rings	177
4.5.4	Factorisation of polynomials over a field	179
4.5.5	Factorisation of polynomials over a factorial ring	182
4.5.6	Polynomials with rational or integer coefficients	188
4.6	Lagrange interpolation and its applications	191
4.7	Kronecker's factorisation method	195
	Appendix to Chapter 4	198
A4	Theoretical exercises	198
B4	Computational exercises	204
C4	Programming exercises	211
5	Finite fields and polynomial congruences	213
5.1	Some field theory	213
5.1.1	Field extensions	213
5.1.2	Algebraic extensions	214
5.1.3	Splitting field of a polynomial	217
5.1.4	Roots of unity	218
5.1.5	Algebraic closure	219
5.1.6	Finite fields and their subfields	220
5.1.7	Automorphisms of finite fields	222
5.1.8	Irreducible polynomials over \mathbb{Z}_p	222
5.1.9	The field \mathbb{F}_4 of order four	224
5.1.10	The field \mathbb{F}_8 of order eight	225
5.1.11	The field \mathbb{F}_{16} of order sixteen	226
5.1.12	The field \mathbb{F}_9 of order nine	226
5.1.13	About the generators of a finite field	227
5.1.14	Complexity of operations in a finite field	228
5.2	Non-linear polynomial congruences	229
5.2.1	Degree two congruences	234
5.2.2	Quadratic residues	236
5.2.3	Legendre symbol and its properties	238
5.2.4	The law of quadratic reciprocity	243
5.2.5	The Jacobi symbol	245
5.2.6	An algorithm to compute square roots	248
	Appendix to Chapter 5	251
A5	Theoretical exercises	251
B5	Computational exercises	255
C5	Programming exercises	260

6	Primality and factorisation tests	261
6.1	Pseudoprime numbers and probabilistic tests	261
6.1.1	Pseudoprime numbers	261
6.1.2	Probabilistic tests and deterministic tests	263
6.1.3	A first probabilistic primality test	263
6.1.4	Carmichael numbers	264
6.1.5	Euler pseudoprimes	265
6.1.6	The Solovay–Strassen probabilistic primality test	268
6.1.7	Strong pseudoprimes	268
6.1.8	The Miller–Rabin probabilistic primality test	272
6.2	Primitive roots	273
6.2.1	Primitive roots and index	278
6.2.2	More about the Miller–Rabin test	279
6.3	A polynomial deterministic primality test	281
6.4	Factorisation methods	290
6.4.1	Fermat factorisation method	291
6.4.2	Generalisation of Fermat factorisation method	292
6.4.3	The method of factor bases	294
6.4.4	Factorisation and continued fractions	299
6.4.5	The quadratic sieve algorithm	300
6.4.6	The ρ method	309
6.4.7	Variation of ρ method	311
	Appendix to Chapter 6	313
A6	Theoretical exercises	313
B6	Computational exercises	315
C6	Programming exercises	317
7	Secrets... and lies	319
7.1	The classic ciphers	319
7.1.1	The earliest secret messages in history	319
7.2	The analysis of the ciphertext	325
7.2.1	Enciphering machines	329
7.3	Mathematical setting of a cryptosystem	330
7.4	Some classic ciphers based on modular arithmetic	334
7.4.1	Affine ciphers	336
7.4.2	Matrix or Hill ciphers	340
7.5	The basic idea of public key cryptography	341
7.5.1	An algorithm to compute discrete logarithms	344
7.6	The knapsack problem and its applications to cryptography ...	345
7.6.1	Public key cipher based on the knapsack problem, or Merkle–Hellman cipher	348
7.7	The <i>RSA</i> system	349
7.7.1	Accessing the <i>RSA</i> system	351
7.7.2	Sending a message enciphered with the <i>RSA</i> system ...	352
7.7.3	Deciphering a message enciphered with the <i>RSA</i> system	354

7.7.4	Why did it work?	356
7.7.5	Authentication of signatures with the <i>RSA</i> system	360
7.7.6	A remark about the security of <i>RSA</i> system	362
7.8	Variants of <i>RSA</i> system and beyond	363
7.8.1	Exchanging private keys	363
7.8.2	ElGamal cryptosystem	364
7.8.3	Zero-knowledge proof: persuading that a result is known without revealing its content nor its proof	365
7.8.4	Historical note	366
7.9	Cryptography and elliptic curves	366
7.9.1	Cryptography in a group	367
7.9.2	Algebraic curves in a numerical affine plane	368
7.9.3	Lines and rational curves	369
7.9.4	Hyperelliptic curves	370
7.9.5	Elliptic curves	372
7.9.6	Group law on elliptic curves	374
7.9.7	Elliptic curves over \mathbb{R} , \mathbb{C} and \mathbb{Q}	380
7.9.8	Elliptic curves over finite fields	381
7.9.9	Elliptic curves and cryptography	384
7.9.10	Pollard's $p - 1$ factorisation method	385
	Appendix to Chapter 7	386
A7	Theoretical exercises	386
B7	Computational exercises	390
C7	Programming exercises	401
8	Transmitting without... fear of errors	405
8.1	Birthday greetings	406
8.2	Taking photos in space or tossing coins, we end up at codes	407
8.3	Error-correcting codes	410
8.4	Bounds on the invariants	413
8.5	Linear codes	419
8.6	Cyclic codes	425
8.7	Goppa codes	429
	Appendix to Chapter 8	436
A8	Theoretical exercises	436
B8	Computational exercises	439
C8	Programming exercises	443
9	The future is already here: quantum cryptography	445
9.1	A first foray into the quantum world: Young's experiment	446
9.2	Quantum computers	449
9.3	Vernam's cipher	451
9.4	A short glossary of quantum mechanics	454
9.5	Quantum cryptography	460
	Appendix to Chapter 9	467

XVI Contents

A9	Theoretical exercises	467
B9	Computational exercises	468
C9	Programming exercises	469
Solution to selected exercises		471
	Exercises of Chapter 1	471
	Exercises of Chapter 2	482
	Exercises of Chapter 3	483
	Exercises of Chapter 4	487
	Exercises of Chapter 5	492
	Exercises of Chapter 6	496
	Exercises of Chapter 7	498
	Exercises of Chapter 8	501
	Exercises of Chapter 9	504
References		507
Index		511