
Contents

Part I Integrable Hamiltonian Hierarchies: Spectral Methods

1	Introduction	3
1.1	Brief Historical overview	3
1.2	Fundamental Properties of the Soliton Equations	7
1.2.1	Solving Nonlinear Cauchy Problems	7
1.2.2	Hierarchies of Soliton Equations	9
1.2.3	Linearized NLEE and Fourier Transform	9
1.2.4	Integrals of Motion	11
1.2.5	Soliton Solutions and Soliton Interactions	11
1.2.6	Hamiltonian Hierarchies	12
1.2.7	Exact Integrability and “Action-Angle” Variables	14
1.2.8	The Hierarchy of Bäcklund and Darboux Transformations	14
1.2.9	Gauge Equivalent Hierarchies	16
1.3	The ISM as the Generalized Fourier Transform	17
1.4	Related Developments, Bibliography and Comments	26
	References	26
2	The Lax Representation and the AKNS Approach	37
2.1	The Lax Representation in the AKNS Approach	37
2.2	The Recursion Operators and the NLEE	41
2.3	Evolution of the Scattering Data	43
2.4	Generalizations of the AKNS Method I	46
2.5	Generalizations of the AKNS Method II	52
2.6	Comments and Bibliographical Review	58
	References	61

3	The Direct Scattering Problem for the Zakharov–Shabat System	71
3.1	Analytic Properties of the Jost Solutions	71
3.2	The Spectrum of L	77
3.3	Asymptotic Behavior for $\lambda \rightarrow \infty$	80
3.4	The Dispersion Relation for $a^\pm(\lambda)$	81
3.5	Minimal Sets of Scattering Data	84
3.6	Spectral Representations of the Jost Solutions	86
3.7	Completeness of the Jost Solutions	88
3.8	Comments and Bibliographical Review	92
	References	93
4	The Inverse Scattering Problem for the Zakharov–Shabat System	97
4.1	Derivation of the GLM Equation	97
4.2	Derivation of GLM Equation from the Completeness Relation (3.111)	100
4.3	The Riemann–Hilbert Problem	103
4.3.1	The Additive RHP for Scalar and Matrix Functions	103
4.3.2	The Multiplicative RHP for Scalar Functions	105
4.3.3	The Multiplicative RHP for Matrix-Valued Functions	107
4.4	The Zakharov–Shabat Dressing Method	109
4.4.1	The RHP and Singular Integral Equations	112
4.4.2	Reflectionless Potentials	113
4.4.3	The 1-Soliton Case	115
4.5	The Dressing Method: The Singular Solutions	116
4.6	Soliton Interactions	120
4.7	Comments and Bibliographical Review	124
	References	127
5	The Generalized Fourier Transforms	133
5.1	The Wronskian Relations	134
5.1.1	The Mapping from q to \mathcal{T}	134
5.1.2	The Mapping from δq to $\delta\mathcal{T}$	137
5.1.3	Still More Wronskian Relations	138
5.2	Completeness of the “Squared” Solutions	139
5.2.1	The Symplectic Basis	143
5.3	Expansions Over the “Squared” Solutions	144
5.3.1	Expansions of $q(x)$	146
5.3.2	Expansions of $\sigma_3\delta q(x)$	147
5.4	Generating Operators Revisited	149
5.5	Spectral Properties of the Λ -Operators	152
5.5.1	The Biquadratic Relations	153
5.5.2	Biorthogonality of the Squared Solutions	156
5.5.3	The Green Functions of Λ_\pm	157

5.6	Expansions Over the “Products of Solutions”	159
5.6.1	Generalized Wronskian Relations	159
5.6.2	Expanding $X(x)$ Over the “Products of Solutions”	164
5.6.3	Generalized Recursion Operators	166
5.7	Comments and Bibliographical Review	168
	References	170
6	Fundamental Properties of the Solvable NLEEs	175
6.1	Description of the Class of NLEEs	175
6.2	Examples of NLEEs	178
6.2.1	Polynomial Dispersion Laws	178
6.2.2	Singular Dispersion Laws	180
6.3	Involutions of the Zakharov–Shabat System	183
6.4	Fundamental Properties of the Soliton Solutions	187
6.4.1	The M -Operators in Terms of A_{\pm}	188
6.4.2	The Trace Identities and A -Operators	190
6.4.3	Recurrent Relations for the Densities of C_p	192
6.4.4	Generating Operators and the Integrals of Motion	196
6.4.5	The Lenard Relation	197
6.5	The Class of Bäcklund Transformations	198
6.6	Comments and Bibliographical Review	203
	References	204
7	Hierarchies of Hamiltonian Structures	211
7.1	Hamiltonian Properties: Basic Examples	211
7.2	Complexified Phase Spaces and Hamiltonians	215
7.3	The Generic NLEEs as Completely Integrable Complex Hamiltonian System	218
7.4	The Hierarchy of Hamiltonian Structures	224
7.5	Involutions and Hierarchies	228
7.5.1	The Involution $q^- = (q^+)^* = u(x)$	228
7.5.2	The Involution $q^- = -(q^+)^* = u(x)$	230
7.5.3	The Involution $q^+ = \pm q^- = u(x)$	231
7.5.4	Applying Both Involutions	234
7.6	Comments and Bibliographical Review	237
	References	238
8	The NLEEs and the Gauge Transformations	247
8.1	The Group of the Gauge Transformations	249
8.2	Gauge-Equivalent NLEE	254
8.3	Direct and Inverse Scattering Problem for \tilde{L}	259
8.4	The Dressing Method and Soliton Solutions	264
8.5	The Wronskian Relations and the Gauge Equivalence	270
8.6	Generalized Fourier Transform and Gauge Transformations	277
8.7	Fundamental Properties of the Gauge-Equivalent NLEEs	282

8.8	The Generic HF-Type NLEE as Completely Integrable Complex Hamiltonian System	288
8.9	Hamiltonian Hierarchies and Gauge Transformations	295
8.10	Involutions and Hierarchies	300
	8.10.1 The Involutions $S_3 = S_3^*$, $S_+ = \pm(S_-)^*$	301
	8.10.2 The Involutions $S^+ = \pm S^-$	304
	8.10.3 Applying Both Involutions	306
8.11	Comments and Bibliographical Review	309
	References	310
9	The Classical r-Matrix Method	315
9.1	The Classical r -Matrix and the NLEE of NLS Type	315
9.2	The Classical r -Matrix and the NLEE of Heisenberg Ferromagnet Type	322
9.3	Jacobi Identity and the Classical Yang–Baxter Equations	326
9.4	The Classical r -Matrix and the Lax Representation	331
9.5	The Classical r -Matrix and the Involutivity of the Integrals of Motion in the Case of Arbitrary Semisimple Lie Algebra	338
9.6	Possibilities for Generalizations of the r -Matrix Approach. A Short Review	343
9.7	Comments and Bibliographical Review	348
	References	349

**Part II Integrable Hamiltonian Hierarchies:
Geometric Theory of the Recursion Operators**

10	Introduction	357
	References	370
11	Smooth Manifolds	373
11.1	Basic Objects: Manifolds, Vector and Tensor Fields	374
11.2	Basic Operations with Tensor Fields	380
11.3	Local Flows	387
11.4	Distributions (Fields of Subspaces)	391
11.5	Related Tensor Fields	394
11.6	Local Form of the Geometric Objects	396
	References	406
12	Hamiltonian Dynamics	407
12.1	Symplectic Structures	407
	12.1.1 Fundamental Fields of Symplectic Form	414
	12.1.2 Restriction of a Symplectic Structure on Submanifold	416
12.2	Real and Complex Hamiltonian Systems	421

12.2.1	Complexified Hamiltonian Systems	421
12.2.2	Real Hamiltonian Forms	428
12.3	Poisson Structures	436
12.3.1	Fundamental Fields of a Poisson Tensor	441
12.3.2	Restriction of Poisson Tensor on Submanifold	441
12.4	Mixed Tensor Fields and Integrability	444
12.4.1	Multi-Hamiltonian Formulations of the Integrable Systems	447
12.4.2	Recursion Operators for Integrable Systems	450
	References	455
13	Vector-Valued Differential Forms	459
13.1	Derivations of the Graded Ring of the Exterior Forms	459
13.2	Vector-Valued Forms of Degree One	466
13.3	Nijenhuis Tensors	468
	References	471
14	Integrability and Nijenhuis Tensors	473
14.1	Integrability Criteria and Nijenhuis Tensors	473
14.2	Recursion Operators in Dissipative Dynamics	477
14.2.1	The Burgers Equation Hierarchy	478
14.3	Noncommutative Integrability Criteria	482
14.3.1	The Invariant Tensor Fields in the Noncommutative Case	485
14.3.2	The Kepler Dynamics	490
14.4	Compatible Poisson Structures and Poisson-Nijenhuis Structures	499
14.5	Principal Properties of Poisson-Nijenhuis Manifolds	507
14.6	Hierarchies of Poisson Structures	509
	References	511
15	Poisson–Nijenhuis Structures Related to the Generalized Zakharov–Shabat System	515
15.1	Poisson–Nijenhuis Structures for GZS System in Canonical Gauge	515
15.2	Poisson–Nijenhuis Structures on Lie Groups and Algebras	520
15.2.1	The Momentum Map	520
15.2.2	Momentum Maps on Lie Groups	524
15.2.3	Algebraic Approaches. Gel’fand–Fuchs Cocycle	529
15.3	Poisson–Nijenhuis Structures on Coadjoint Orbits	532
15.3.1	The Manifold of Jost Solutions	532
15.3.2	The Manifold of Potentials in Pole Gauge	537
15.4	Fundamental Fields for GZS System	541
	References	544

16 Linear Bundles of Lie Algebras and Compatible Poisson Structures	547
16.1 Preliminaries	547
16.2 General Properties	549
16.3 Construction of Closed Linear Bundles of Lie Algebras	556
16.4 Poisson-Lie Tensors Over the Algebras \mathfrak{g}_S	562
16.5 Finite Dimensional Applications	564
16.5.1 The Clebsh and the Neumann System	564
16.5.2 The Algebra $\mathfrak{o}(4)$	569
16.6 The Chiral Fields Hierarchy and the Associated Recursion Operators	578
16.7 Polynomial Lax Representation of the Landau–Lifshitz Hierarchy	582
16.8 Recursion Operators for LLp Hierarchy	589
16.9 LLp Hierarchy – the Poisson-Nijenhuis Structure	596
16.10 Chiral Fields Hierarchy : the Poisson-Nijenhuis Structure ...	601
References	610
Appendix: Generalizations	613
A.1 Algebraic Approach to Integrability	613
A.1.1 Algebraic Differentiable Calculus	614
A.1.2 Poisson Rings, Bi-Hamiltonian Systems and Nijenhuis Tensors	617
A.2 Integrability of Dynamics with Fermionic Variables	624
A.2.1 Graded Differential Calculus	625
A.2.2 Poisson Supermanifolds and Super Nijenhuis Tensors .	628
References	633
Abbreviations	635
Index	637